Math 2040 Tutorial Week 2

1.2 Vector Space

8. In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in V$.

Solution: By axioms of vector spaces,

$$(a+b)(x+y) = a(x+y) + b(x+y) = ax + ay + bx + by$$

1.3 Subspaces

11. Is the set $W = \{ f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n \}$ a subspeece of $P(F)$ if $n \geq 1$?

Solution: No if $n \neq 1$. W is NOT closed under addition: For example,

When $n=2$, $(x^2+x) + (-x^2) = x \notin W$

Yes if $n = 1$.

1.4 Linear Combination

10. Show that if $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2x2 matrices

Solution: $\forall A \in M_{2\times 2}(F)$ can be written as $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ $a,b,c \in F$

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{span}\{M_1, M_2, M_3\}$$

1.5 Linear Dependence

9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of each other.

Solution:

" $\Rightarrow$ " u v are l.d. $\Leftrightarrow$ $au+bv=0$ for some $a, b \in F$. and at least one of a and b is not 0, say a. Then $u = -\frac{b}{a}v$.

" $\Leftarrow$ " If $u = tv$ $t \in F$, then $u - tv = 0$.

10. Give an example of 3 linearly dependent vectors in R^3 such that none of the three is a multiple of each other.

Solution: $v_1 = (1, 1, 0)$ $v_2 = (1, 0, 0)$ $v_3 = (0, 1, 0)$

$v_1 - v_2 - v_3 = 0$.

1.6 Bases and Dimension

11. Let u, v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a, b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.

Solution: If $\{u, v\}$ is a basis, then $\dim V = 2$. We only have to check that $\{u+v, au\}$ $\{au, bv\}$ are lin. ind.

Assuming $s(u+v) + t \cdot au = 0$ $\Leftrightarrow$ $(s+ta)u + sv = 0$ we have $s+ta = s = 0$

$\Rightarrow$ $s = t = 0$ $\Rightarrow$ $\{u+v, au\}$ lin. ind.

$s \cdot au + t \cdot bv = 0$ we have $sa = tb = 0$ or $s = t = 0$. $\Rightarrow$

$\{au, bv\}$ lin ind.

2.2 Matrix Rep'n

3. $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. β is the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$ compute $[T]_\alpha^\gamma$.

Solution: $T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0\cdot(0,1,1) + \frac{2}{3}(2,2,3)$

$T(0,1) = (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3)$

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

$T(1,2) = (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$T(2,3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

3.4 Linear Equations

5. Let the reduced row echelon form of A be $\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$. If the 1st, 2nd, and 4th columns of A are $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$. Determine A.

Solution: $B := \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} = (b_1 \cdots b_5)$

$b_3 = 2e_1 - 5e_2 \qquad b_5 = -2e_1 - 3e_2 + 6e_3$

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$

4.2 Determinant

5. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the 1st row

Solution: $\det = 0 \cdot \det\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} - 1 \cdot \det\begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$

$= -12$

7. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the 2nd row

Solution. $\det = 1 \cdot \det\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = -12$

2.1 Linear Transformations

12. Is there a linear transformation $T: R^3 \to R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$

Solution: No. $T(-2,0,-6) = -2(T(1,0,3)) = (2,2) \neq (2,1)$

15. $T: P(\mathbb{R}) \to P(\mathbb{R})$ by $T(f(x)) = \int_0^x f(t)\,dt$ Prove that T is linear and one to one, but not onto.

Solution: $T(f(x)+g(x)) = \int_0^x (f(t)+g(t))\,dt = T(f(x)) + T(g(x))$

$T(c f(x)) = \int_0^x c \cdot f(t)\,dt = c \cdot T(f(x))$

If $T(f(x)) = T(g(x))$, then $\int_0^x (f(t)-g(t))\,dt = 0$.

$f(t)$, $g(t)$ are polynomials, by calculus, we know that $f(x)=g(x)$

$\{x, x^2 \cdots\}$ is a basis for $R(T)$.

Span $R(T) \neq P(\mathbb{R}) \Rightarrow T$ is not onto.

17. V, W are finite dimensional vector spaces. $T: V \to W$ is linear.

a) Prove that if $\dim V < \dim W$, then T cannot be onto.

b) Prove that if $\dim V > \dim W$, then T cannot be 1-1.

Solution: a) By Thm 2.3 $\dim R(T) \leq \dim V$. By hypothesis, $\dim V < W$. $\Rightarrow \dim R(T) < W \Rightarrow T$ cannot be onto.

b) If T is 1-1, then nullity$(T) = 0$, then $\dim V = \text{rank } T > \dim W$

Contradiction with $\dim R(T) \leq \dim W$

2.2 Matrix Rep'n

9. V is the vector space of complex numbers over $\mathbb{R}$. Define $T: V \to V$ by $T(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$.

Solution: $\forall\, x = a + bi,\ y = c + di \in \mathbb{C},\ a, b, c, d \in \mathbb{R}$.

$$T(\alpha x + y) = \cdots = (\alpha a - \alpha b i) + (c - d i) = \alpha T(x) + T(y)$$

$\Rightarrow T$ is linear.

$$T(1) = 1 = 1 \cdot 1 + 0 \cdot i \qquad T(i) = -i = 0 \cdot 1 + (-1) \cdot i$$

$$A = [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

13. V, W are vector spaces. T, U are nonzero linear transformation from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is linear independent subset of $\mathcal{L}(V, W)$.

Solution: We want to show that if $aT + bU = 0$, then $a = b = 0$, $a, b \in F$.

$$aT(v) + bU(v) = (aT + bU)(v) = 0 \Rightarrow (aT)(v) = -(bU)(v) \in R(T) \cap R(U)$$

$$\Rightarrow aT(v) = bU(v) = 0 \quad \forall v \in V.$$

$$\Rightarrow aT = bU = 0 \quad \Rightarrow \quad a = 0,\ b = 0$$

why? – Check directly!